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INCREMENTAL DEFORMATION MODEL FOR A ROD

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A nonlinear deformation model for a rod with rigid cross sections is proposed. A complete system of local incremental equations, a variational equation equivalent to this system, and an equation of virtual work are formulated. Numerical analysis of the deformation of a ring transmission is performed.

Formulations of the problems of deformable solid mechanics in increments of the desired functions are used to calculate the increments of the deformation parameters in transition from one deformed state to another state that is rather close to the former. This approach leads to a constructive method for analyzing nonlinear problems since it reduces each of these problems to a sequence of linear problems. This approach has no alternative in the case where the constitutive equations of material are given in the form of relations between stress and strain increments.

1. Equations of Finite Deformation of a Rod. We consider a rod as a solid body in a three-dimensional Cartesian space. The rod material is distributed over a small neighborhood G of a certain (base) line $C_3 \subset G$. A system of curvilinear coordinates t_J is related to this line in such a manner that t_3 is an internal parameter of the line, t_1 and t_2 are the transverse coordinates orthogonal to t_3, $t_i \in A$, $t_3 \in [l_1, l_2]$, A is an arbitrary cross section of the rod, and l_i is a real number.

In this paper, we use the notation adopted in [1, 2]. Variations of the deformation parameters are denoted by the symbol δ, and the desired and specified increments are denoted by the symbol Δ. The capital Latin subscripts take values 1, 2, and 3, and the lower-case Latin subscripts take values 1 and 2. The rule of summation over repeated indices is employed. Possible dependence on time is not indicated explicitly.

In the three-dimensional space above G, we define the position vector $\boldsymbol{g}(t_J \in G)$ of an arbitrary point of the rod, the position vector $\boldsymbol{a}(t_J \in C_3)$ of an arbitrary point on the base line, and the local coordinate basis $\boldsymbol{a}_J(\boldsymbol{a})$ related to points on the line and consisting of the tangent vector $\boldsymbol{a}_3$ and the vectors $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ orthogonal to the tangent vector.

The rod is defined by the equation $\boldsymbol{g} = \boldsymbol{a} + t_i\boldsymbol{a}_i$. The equalities $\boldsymbol{g}_J \equiv \partial_J\boldsymbol{g}$, $\boldsymbol{g}_i = \boldsymbol{a}_i$, $\boldsymbol{g}_3 = \boldsymbol{a}_3 + t_i\boldsymbol{b}_i$, $\boldsymbol{a}_3 \equiv \partial_3\boldsymbol{a}$, and $\boldsymbol{b}_i \equiv \partial_3\boldsymbol{a}_i$ (∂_J denotes differentiation with respect to t_J) introduce the body basis $\boldsymbol{g}_J(\boldsymbol{g})$ of the coordinate system and express this basis through the contour basis $\boldsymbol{a}_J(\boldsymbol{a})$, which, without loss of generality, can be considered orthonormal.

Deformation of the rod into a certain finite state is represented by the mapping $\boldsymbol{g} \to \boldsymbol{g}^+(\boldsymbol{g}), \boldsymbol{g}_J \to \boldsymbol{g}_J^+(\boldsymbol{g})$ and $\boldsymbol{g}_J^+ \equiv \partial_J\boldsymbol{g}^+$. The base line and its basis deform together with the rod: $\boldsymbol{a} \to \boldsymbol{a}^+(\boldsymbol{a})$ and $\boldsymbol{a}_J \to \boldsymbol{a}_J^+(\boldsymbol{a})$. The local orthogonal transformation

$$\boldsymbol{a}_J^0 = \boldsymbol{\Theta} \cdot \boldsymbol{a}_J, \qquad \partial_i\boldsymbol{\Theta} \equiv \boldsymbol{0}, \qquad \boldsymbol{\Theta} \cdot \bar{\boldsymbol{\Theta}} \equiv \boldsymbol{1} \tag{1.1}$$

with rotation tensor $\boldsymbol{\Theta}(\boldsymbol{a})$ introduces the *convective* basis $\boldsymbol{a}_J^0(\boldsymbol{a})$ with initial value $\boldsymbol{a}_J(\boldsymbol{a})$. Henceforth, this basis is assumed to be the determining basis for the vector spaces above G and C_3. Transformation (1.1) is represented in the form transposed with respect to that in [2], which is more common in the matrix calculus.

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The finite deformed state of the rod is described by the equation

$$\boldsymbol{g}^+ = \boldsymbol{a}^+ + t_i \boldsymbol{a}_i^0. \tag{1.2}$$

This leads to the equalities $\boldsymbol{g}_i^+ = \boldsymbol{a}_i^+ = \boldsymbol{a}_i^0$, which identify the deformed transverse vectors with the convective vectors. Generally, the vectors $\boldsymbol{g}_3^+$, $\boldsymbol{a}_3^+$, and $\boldsymbol{a}_3^0$ do not coincide with each other. Since the basis $\boldsymbol{a}_J^0$ is orthogonal by the definition and the basis $\boldsymbol{a}_J^+$ can be nonorthogonal, the convective vector $\boldsymbol{a}_3^0$, unlike $\boldsymbol{a}_3^+$, is no longer tangent to the deformed base line, i.e., $\boldsymbol{a}_3^0 \neq \boldsymbol{a}_3^+$ both in magnitude and direction.

Equation (1.2) corresponds to the linear approximation of the body displacement field $\boldsymbol{w}(\boldsymbol{g}) \equiv \boldsymbol{g}^+ - \boldsymbol{g}$ with respect to the transverse coordinates

$$\boldsymbol{w} = \boldsymbol{u} + t_i \boldsymbol{v}_i, \qquad \boldsymbol{u} \equiv \boldsymbol{a}^+ - \boldsymbol{a}, \qquad \boldsymbol{v}_i \equiv \boldsymbol{a}_i^0 - \boldsymbol{a}_i. \tag{1.3}$$

These relations specify rigid-body motion of a cross section of the rod with translational displacement $\boldsymbol{u}(\boldsymbol{a})$ and rotation $t_i \boldsymbol{v}_i(\boldsymbol{a})$.

The body strain field of the rod is defined by the vectors $\boldsymbol{w}_I(\mathbf{g})$:

$$\boldsymbol{w}_I \equiv \boldsymbol{g}_I^+ - \boldsymbol{\Theta} \cdot \boldsymbol{g}_I = \partial_I \boldsymbol{w} - (\boldsymbol{\Theta} - \mathbf{1}) \cdot \boldsymbol{g}_I. \tag{1.4}$$

Using approximations (1.2) and (1.3), we obtain the equalities

$$\boldsymbol{w}_3 = \boldsymbol{u}_3 + t_i \boldsymbol{v}_{3i}, \qquad \boldsymbol{w}_i \equiv \mathbf{0}, \tag{1.5}$$

which express the body field in terms of the contour vectors $\boldsymbol{u}_3(\boldsymbol{a})$ and $\boldsymbol{v}_{3i}(\boldsymbol{a})$ of metric and torsional–flexural deformations:

$$\begin{gathered} \boldsymbol{u}_3 \equiv \boldsymbol{a}_3^+ - \boldsymbol{\Theta} \cdot \boldsymbol{a}_3 = \partial_3 \boldsymbol{u} - (\boldsymbol{\Theta} - 1) \cdot \boldsymbol{a}_3, \qquad \boldsymbol{a}_3^+ \equiv \partial_3 \boldsymbol{a}^+, \\ \boldsymbol{v}_{3i} \equiv \boldsymbol{b}_i^+ - \boldsymbol{\Theta} \cdot \boldsymbol{b}_i = \partial_3 \boldsymbol{\Theta} \cdot \boldsymbol{a}_i, \qquad \boldsymbol{b}_i^+ \equiv \partial_3 \boldsymbol{a}_i^0. \end{gathered} \tag{1.6}$$

The last formulas define these in terms of *primary* unknowns — the displacement vector $\boldsymbol{u}(\boldsymbol{a})$ and the rotation tensor $\boldsymbol{\Theta}(\boldsymbol{a})$.

For a deformed state, the local dynamic equations

$$\partial_3 \boldsymbol{x}^3 + \boldsymbol{p} = \mathbf{0}, \quad \partial_3 \boldsymbol{y}^3 + \boldsymbol{x} + \boldsymbol{q} = \mathbf{0}, \quad \boldsymbol{x} \equiv \boldsymbol{a}_3^+ \times \boldsymbol{x}^3, \quad \boldsymbol{y}^3 \equiv \boldsymbol{a}_i^0 \times \boldsymbol{y}_i^3 \tag{1.7}$$

hold on C_3. If at a certain point $t_3 = l_\lambda$ the forces and moments are specified by the vectors $\boldsymbol{p}_\lambda$ and $\boldsymbol{q}_\lambda$, the point dynamic conditions

$$e_{\lambda 3} \boldsymbol{x}^3 - \boldsymbol{p}_\lambda = \mathbf{0}, \qquad e_{\lambda 3} \boldsymbol{y}^3 - \boldsymbol{q}_\lambda = \mathbf{0} \tag{1.8}$$

are satisfied there. At the point $t_3 = l_\mu$,

$$\boldsymbol{u} = \boldsymbol{u}_\mu, \qquad \boldsymbol{\Theta} = \boldsymbol{\Theta}_\mu, \tag{1.9}$$

where $\boldsymbol{u}_\mu$ and $\boldsymbol{\Theta}_\mu$ are specified values of the displacement vector and the rotation tensor, respectively.

The unknown vector functions $\boldsymbol{x}^3(\boldsymbol{a})$ and $\boldsymbol{y}_i^3(\boldsymbol{a})$ in Eqs. (1.7) are the mathematical moments of the stress vector $\boldsymbol{z}^3(\boldsymbol{g})$ over the cross section of the rod:

$$\boldsymbol{x}^3 \equiv \int\limits_A \boldsymbol{z}^3 J dA, \qquad \boldsymbol{y}_i^3 \equiv \int\limits_A \boldsymbol{z}^3 t_i J dA. \tag{1.10}$$

The functions $\boldsymbol{x}^3(\boldsymbol{a})$ and $\boldsymbol{y}^3(\boldsymbol{a})$ have the mechanical meaning of the principal vector and the principal moment.

Generally, Eqs. (1.7) cannot be solved for the force factors $\boldsymbol{x}^3$ and $\boldsymbol{y}^3$ for the following reasons: 1) the vector $\boldsymbol{a}_3^+$ depends on the displacement field; 2) in dynamic problems, the vectors of external loads $\boldsymbol{p}$ and $\boldsymbol{q}$ always depend on the displacements and rotations, since they include inertial forces and in static problems, these dependences arise when external loads change during deformation. In the general problem of the finite deformation of a rod, the dynamic equations (1.7) and (1.8) are combined with the kinematic equations (1.6) and (1.9) and constitutive equations that relate the dynamic vectors $\boldsymbol{x}^3$ and $\boldsymbol{y}^3$ and the kinematic vectors $\boldsymbol{u}_3$ and $\boldsymbol{v}_{3i}$. A finite formulation of these relations is possible only in particular cases. An incremental formulation is more general. This naturally leads to the necessity of constructing an incremental deformation model for the rod.

2. Formulation of Local Equations of the Model. In deriving the equations of deformation of the rod, we use the following variation rules for vector and tensor fields:

$$\delta\Theta = \delta\boldsymbol{\Omega}\cdot\Theta, \quad \delta\boldsymbol{\Omega} = \delta\boldsymbol{\omega}\times\mathbf{1} = \mathbf{1}\times\delta\boldsymbol{\omega}, \quad \delta\boldsymbol{a}_J^0 = \delta\boldsymbol{\Omega}\cdot\boldsymbol{a}_J^0 = \delta\boldsymbol{\omega}\times\boldsymbol{a}_J^0, \quad \delta\boldsymbol{a}_3^+ = \partial_3\delta\boldsymbol{u},$$
$$\delta_0\boldsymbol{u}_3 = \partial_3\delta\boldsymbol{u} - \delta\boldsymbol{\Omega}\cdot\boldsymbol{a}_3^+ = \partial_3\delta\boldsymbol{u} - \delta\boldsymbol{\omega}\times\boldsymbol{a}_3^+, \quad \delta_0\boldsymbol{v}_{3i} = \partial_3\delta\boldsymbol{\Omega}\cdot\boldsymbol{a}_i^0 = \partial_3\delta\boldsymbol{\omega}\times\boldsymbol{a}_i^0. \tag{2.1}$$

Here $\delta\boldsymbol{\Omega}(\boldsymbol{a})$ and $\delta\boldsymbol{\omega}(\boldsymbol{a})$ are the spin and vector of virtual rotation, and δ_0 is a relative variation operator such that $\delta_0\boldsymbol{a}_J^0 \equiv 0$ and for any vector $\boldsymbol{v}$ specified in the convective basis, the equalities

$$\delta\boldsymbol{v} = \delta_0\boldsymbol{v} + \delta\boldsymbol{\Omega}\cdot\boldsymbol{v} = \delta_0\boldsymbol{v} + \delta\boldsymbol{\omega}\times\boldsymbol{v} \tag{2.2}$$

hold by definition. Formulas (2.1) contain two primary virtual vectors $\delta\boldsymbol{u}$ and $\delta\boldsymbol{\omega}$. In vector products the latter can be replaced by the spin tensor $\delta\boldsymbol{\Omega}$.

Using (1.7), we write the incremental dynamic equations on C_3 as

$$\partial_3\Delta\boldsymbol{x}^3 + \Delta\boldsymbol{p} = \mathbf{0}, \qquad \partial_3\Delta\boldsymbol{y}^3 + \Delta\boldsymbol{x} + \Delta\boldsymbol{q} = \mathbf{0}. \tag{2.3}$$

These are supplemented by the following dynamic and kinematic conditions at the points l_λ and l_μ:

$$e_{\lambda 3}\Delta\boldsymbol{x}^3 - \Delta\boldsymbol{p}_\lambda = \mathbf{0}, \qquad e_{\lambda 3}\Delta\boldsymbol{y}^3 - \Delta\boldsymbol{q}_\lambda = \mathbf{0}; \tag{2.4}$$

$$\Delta\boldsymbol{u} = \Delta\boldsymbol{u}_\mu, \qquad \Delta\boldsymbol{\omega} = \Delta\boldsymbol{\omega}_\mu. \tag{2.5}$$

The dynamic variables are related to the kinematic variables by constitutive relations. For elastic and elastoplastic deformation of the rod in the region G, these can be expressed by the equation

$$\Delta_0\boldsymbol{z}^3 = \boldsymbol{D}\cdot\Delta_0\boldsymbol{w}_3, \tag{2.6}$$

where Δ_0 is the relative increment operator defined similarly to δ_0 and $\boldsymbol{D}$ is the dyadic tensor of the material stiffness that takes the prehistory of loading into account.

From (1.10) and (2.6) we have constitutive relations on C_3 for the contour variables

$$\Delta_0\boldsymbol{x}^3 = \boldsymbol{E}\cdot\Delta_0\boldsymbol{u}_3 + \boldsymbol{E}_j\cdot\Delta_0\boldsymbol{v}_{3j}, \qquad \Delta_0\boldsymbol{y}_i^3 = \boldsymbol{E}_i\cdot\Delta_0\boldsymbol{u}_3 + \boldsymbol{E}_{ij}\cdot\Delta_0\boldsymbol{v}_{3j} \tag{2.7}$$

with the generalized stiffness tensors $\boldsymbol{E} \equiv \int\limits_A \boldsymbol{D}J\,dA$, $\boldsymbol{E}_i \equiv \int\limits_A \boldsymbol{D}t_iJ\,dA$, and $\boldsymbol{E}_{ij} \equiv \int\limits_A \boldsymbol{D}t_it_jJ\,dA$. The vectors $\Delta_0\boldsymbol{u}_3$ and $\Delta_0\boldsymbol{v}_{3i}$ in (2.7) are calculated according to the variation rules (2.1):

$$\Delta_0\boldsymbol{u}_3 = \partial_3\Delta\boldsymbol{u} - \Delta\boldsymbol{\Omega}\cdot\boldsymbol{a}_3^+, \qquad \Delta_0\boldsymbol{v}_{3i} = \partial_3\Delta\boldsymbol{\Omega}\cdot\boldsymbol{a}_i^0 = \partial_3\Delta\omega\times\boldsymbol{a}_i^0. \tag{2.8}$$

Using (2.8) and the equality $\Delta_0\boldsymbol{y}^3 = \boldsymbol{a}_i^0\times\Delta_0\boldsymbol{y}_i^3$, which is valid by definition, we bring Eqs. (2.7) to the form

$$\Delta_0\boldsymbol{x}^3 = \boldsymbol{E}\cdot\Delta_0\boldsymbol{u}_3 + \boldsymbol{F}\cdot\partial_3\Delta\omega, \qquad \Delta_0\boldsymbol{y}^3 = \boldsymbol{G}\cdot\Delta_0\boldsymbol{u}_3 + \boldsymbol{H}\cdot\partial_3\Delta\omega \tag{2.9}$$

where $\boldsymbol{F} \equiv -\boldsymbol{E}_j\times\boldsymbol{a}_j^0$, $\boldsymbol{G} \equiv \boldsymbol{a}_i^0\times\boldsymbol{E}_i$, and $\boldsymbol{H} \equiv -\boldsymbol{a}_i^0\times(\boldsymbol{E}_{ij}\times\boldsymbol{a}_j^0)$ are the modified stiffness tensors. Here and below, we assume that relations (2.9) admit the inversion

$$\partial_3\Delta\boldsymbol{u} - \Delta\boldsymbol{\Omega}\cdot\boldsymbol{a}_3^+ = \tilde{\boldsymbol{E}}\cdot\Delta_0\boldsymbol{x}^3 + \tilde{\boldsymbol{F}}\cdot\Delta_0\boldsymbol{y}^3, \qquad \partial_3\Delta\omega = \tilde{\boldsymbol{G}}\cdot\Delta_0\boldsymbol{x}^3 + \tilde{\boldsymbol{H}}\cdot\Delta_0\boldsymbol{y}^3 \tag{2.10}$$

with known compliance tensors $\tilde{\boldsymbol{E}}$, $\tilde{\boldsymbol{F}}$, $\tilde{\boldsymbol{G}}$, and $\tilde{\boldsymbol{H}}$.

Equations (2.3)–(2.5) and (2.9) or (2.10) form a complete system of local equations for increments of the desired functions.

3. Variational Formulation of the Problem. In the functional space $L_2(C_3)$, we introduce arbitrary variations $\delta\boldsymbol{u}$, $\delta\boldsymbol{\omega}$, $\delta\boldsymbol{x}^3$, and $\delta\boldsymbol{y}^3$ of the kinematic and dynamic vectors. We replace the local equations (2.3)–(2.5) and (2.10) by the Galerkin integral equality

$$\int\limits_{C_3}((\partial_3\Delta\boldsymbol{x}^3 + \Delta\boldsymbol{p})\cdot\delta\boldsymbol{u} + (\partial_3\Delta\boldsymbol{y}^3 + \Delta\boldsymbol{x} + \Delta\boldsymbol{q})\cdot\delta\boldsymbol{\omega})\,dt_3$$

$$+\int_{C_3}(\tilde{\boldsymbol{E}}\cdot\Delta_0\boldsymbol{x}^3+\tilde{\boldsymbol{F}}\cdot\Delta_0\boldsymbol{y}^3-\partial_3\Delta\boldsymbol{u}+\Delta\boldsymbol{\Omega}\cdot\boldsymbol{a}_3^+)\cdot\delta\boldsymbol{x}^3\,dt_3+\int_{C_3}(\tilde{\boldsymbol{G}}\cdot\Delta_0\boldsymbol{x}^3+\tilde{\boldsymbol{H}}\cdot\Delta_0\boldsymbol{y}^3-\partial_3\Delta\boldsymbol{\omega})\cdot\delta\boldsymbol{y}^3\,dt_3$$

$$+((\Delta\boldsymbol{p}_\lambda-e_{\lambda3}\Delta\boldsymbol{x}^3)\cdot\delta\boldsymbol{u}+(\Delta\boldsymbol{q}_\lambda-e_{\lambda3}\Delta\boldsymbol{y}^3)\cdot\delta\boldsymbol{\omega})_{t_3=l_\lambda}$$

$$+e_{\mu3}((\Delta\boldsymbol{u}-\Delta\boldsymbol{u}_\mu)\cdot\delta\boldsymbol{x}^3+(\Delta\boldsymbol{\omega}-\Delta\boldsymbol{\omega}_\mu)\cdot\delta\boldsymbol{y}^3)_{t_3=l_\mu}=0. \tag{3.1}$$

After integration of the first integral by parts, equality (3.1) takes the form

$$\int_{C_3}(\Delta\boldsymbol{p}\cdot\delta\boldsymbol{u}-\Delta\boldsymbol{x}^3\cdot\partial_3\delta\boldsymbol{u}+(\Delta\boldsymbol{q}+\Delta\boldsymbol{x})\cdot\delta\boldsymbol{\omega}-\Delta\boldsymbol{y}^3\cdot\partial_3\delta\boldsymbol{\omega})\,dt_3$$

$$+\int_{C_3}(\tilde{\boldsymbol{E}}\cdot\Delta_0\boldsymbol{x}^3+\tilde{\boldsymbol{F}}\cdot\Delta_0\boldsymbol{y}^3-\partial_3\Delta\boldsymbol{u}+\Delta\boldsymbol{\Omega}\cdot\boldsymbol{a}_3^+)\cdot\delta\boldsymbol{x}^3\,dt_3+\int_{C_3}(\tilde{\boldsymbol{G}}\cdot\Delta_0\boldsymbol{x}^3+\tilde{\boldsymbol{H}}\cdot\Delta_0\boldsymbol{y}^3-\partial_3\Delta\boldsymbol{\omega})\cdot\delta\boldsymbol{y}^3\,dt_3$$

$$+(\Delta\boldsymbol{p}_\lambda\cdot\delta\boldsymbol{u}+\Delta\boldsymbol{q}_\lambda\cdot\delta\boldsymbol{\omega})_{t_3=l_\lambda}+e_{\mu3}(\Delta\boldsymbol{x}^3\cdot\delta\boldsymbol{u}+\Delta\boldsymbol{y}^3\cdot\delta\boldsymbol{\omega}$$

$$+(\Delta\boldsymbol{u}-\Delta\boldsymbol{u}_\mu)\cdot\delta\boldsymbol{x}^3+(\Delta\boldsymbol{\omega}-\Delta\boldsymbol{\omega}_\mu)\cdot\delta\boldsymbol{y}^3)_{t_3=l_\mu}=0. \tag{3.2}$$

This form requires smoothness of the variation $\delta\boldsymbol{u}$ and $\delta\boldsymbol{\omega}$ along the base line.

For sufficiently smooth integrands, equalities (3.1) and (3.2) are equivalent and, therefore, the following statement is valid.

Statement. *If the vectors $\Delta\boldsymbol{x}^3$, $\Delta\boldsymbol{y}^3$, $\Delta\boldsymbol{u}$, and $\Delta\boldsymbol{\omega}$ are an exact solution of the system of local equations* (2.3)–(2.5) *and* (2.10), *the integral equality* (3.2) *holds for any variations; if certain vectors $\Delta\boldsymbol{x}^3$, $\Delta\boldsymbol{y}^3$, $\Delta\boldsymbol{u}$, and $\Delta\boldsymbol{\omega}$ identically satisfy the integral equality* (3.2) *for any variations, these vectors are an exact solution of the above-mentioned system.*

For the desired integrands of insufficient smoothness, the variational equation (3.2) gives a weak formulation of the problem of rod deformation. In this Galerkin formulation, the smoothness requirements are minimal: the vectors $\Delta\boldsymbol{x}^3$, $\Delta\boldsymbol{y}^3$, $\delta\boldsymbol{x}^3$, and $\delta\boldsymbol{y}^3$ are elements of the space $L_2(C_3)$ and the vectors $\Delta\boldsymbol{u}$, $\Delta\boldsymbol{\omega}$, $\delta\boldsymbol{u}$, and $\delta\boldsymbol{\omega}$ are elements of the Sobolev space $W_2^1(C_3)$.

An important consequence of (3.2) is the following equation for the virtual work of the rod:

$$\int_{C_3}(\Delta\boldsymbol{p}\cdot\delta\boldsymbol{u}-\Delta\boldsymbol{x}^3\cdot\partial_3\delta\boldsymbol{u}+(\Delta\boldsymbol{q}+\Delta\boldsymbol{x})\cdot\delta\boldsymbol{\omega}-\Delta\boldsymbol{y}^3\cdot\partial_3\delta\boldsymbol{\omega})\,dt_3+(\Delta\boldsymbol{p}_\lambda\cdot\delta\boldsymbol{u}+\Delta\boldsymbol{q}_\lambda\cdot\delta\boldsymbol{\omega})_{t_3=l_\lambda}=0. \tag{3.3}$$

It is valid for kinematically possible variations $\delta\boldsymbol{u}$ and $\delta\boldsymbol{\omega}$ such that $\delta\boldsymbol{u}=\delta\boldsymbol{\omega}=\boldsymbol{0}$ at the point $t_3=l_\mu$, and with satisfaction of the local equations (2.10) and point conditions (2.5). Equality (3.3) gives a weak form of the dynamic equations (2.3) and the point conditions (2.4). When the variables $\Delta\boldsymbol{x}^3$ and $\Delta\boldsymbol{y}^3$ are eliminated from (3.3) using equalities (2.9), it takes the meaning of a weak formulation of the problem relative to the kinematic variables $\Delta\boldsymbol{u}$ and $\Delta\boldsymbol{\omega}$ with the principal point conditions (2.5).

For numerical analysis, it is necessary to have a matrix formulation of the variational equation (3.2). The important role of the relative increments in the local equations (2.7)–(2.10) should be pointed out. Precisely this fact determines the choice of the convective basis as the main basis for the vector fields under consideration. Bearing this in mind, we introduce the following decompositions of the desired functions $\Delta\boldsymbol{u}$, $\Delta\boldsymbol{\omega}$, $\Delta_0\boldsymbol{x}^3$, and $\Delta_0\boldsymbol{y}^3$

$$\Delta\boldsymbol{u}=\boldsymbol{a}_J^0\Delta U^J,\qquad \Delta\boldsymbol{\omega}=\boldsymbol{a}_J^0\Delta\Omega^J,\qquad \Delta_0\boldsymbol{x}^3=\boldsymbol{a}_J^0\Delta X^{3J},\qquad \Delta_0\boldsymbol{y}^3=\boldsymbol{a}_J^0\Delta Y^{3J},$$

which are similar to those of the variations $\delta\boldsymbol{u}$, $\delta\boldsymbol{\omega}$, $\delta\boldsymbol{x}^3$, and $\delta\boldsymbol{y}^3$. The increment of any vector $\boldsymbol{v}=\boldsymbol{a}_J^0V^J$ that is different from the primary vectors is calculated according to (2.2): $\Delta\boldsymbol{v}=\Delta_0\boldsymbol{v}+\Delta\boldsymbol{\Omega}\cdot\boldsymbol{v}$. Here $\Delta_0\boldsymbol{v}\equiv\boldsymbol{a}_J^0\Delta V^J$ is the relative increment and $\Delta\boldsymbol{\Omega}$ is the spin of the rotation vector increment.

Differentiation of the convective-basis vectors can be expressed by the transformation

$$\partial_3\boldsymbol{a}_J^0=\boldsymbol{C}_3^0\cdot\boldsymbol{a}_J^0,\qquad \boldsymbol{C}_3^0\equiv(\partial_3\boldsymbol{a}_J^0)\boldsymbol{a}_0^J$$

with the spin tensor $\boldsymbol{C}_3^0(\boldsymbol{a})$. Differentiation of any vector $\boldsymbol{v}(\boldsymbol{a})$ specified in the convective basis is performed by the formula

$$\partial_3 \boldsymbol{v} = \partial_3^0 \boldsymbol{v} + \boldsymbol{C}_3^0 \cdot \boldsymbol{v}, \qquad \partial_3^0 \boldsymbol{v} \equiv \boldsymbol{a}_J^0 \partial_3 V^J,$$

where $\partial_3^0 \boldsymbol{v}$ is the relative derivative of the vector with respect to t_3. The definition of the tensor $\boldsymbol{C}_3^0$ leads to the formula $\Delta_0 \boldsymbol{C}_3^0 = \partial_3 \Delta \boldsymbol{\Omega}$ for its relative increment.

The above rules for variation and differentiation of vector fields are used to obtain a matrix form of the variational equation (3.2). Moreover, the relative increments and derivatives of vectors are basic functions in the equation since they are represented by matrices of increments of the vector components.

The desired functions are calculated by the conventional procedure of successive approximations, which allows one to trace the process of deformation of the rod step by step from the initial (unstressed) state to the final state corresponding to the specified external forces. The initial values of the parameters are specified by the equalities $\boldsymbol{a}_J^+ = \boldsymbol{a}_J^0 = \boldsymbol{a}_J$, $\boldsymbol{b}_i^+ = \boldsymbol{b}_i \equiv \partial_3 \boldsymbol{a}_i$, $\boldsymbol{b}_i^+ = \boldsymbol{b}_i \equiv \partial_3 \boldsymbol{a}_i$, $\boldsymbol{C}_3^0 = \boldsymbol{C}_3 \equiv (\partial_3 \boldsymbol{a}_J)\boldsymbol{a}^J$, $\Theta \equiv \mathbf{1}$, and $\boldsymbol{x}^3 \equiv \boldsymbol{y}^3 \equiv \boldsymbol{p} \equiv \boldsymbol{q} \equiv \boldsymbol{p}_\lambda \equiv \boldsymbol{q}_\lambda \equiv \mathbf{0}$. The material stiffness tensor $\boldsymbol{D}$ is given initially by the Hooke's matrix. If it depends on strains, in the next step, it is introduced by the matrix $[\boldsymbol{D} + \Delta_0 \boldsymbol{D}]$, where $\Delta_0 \boldsymbol{D}$ is the relative increment of the tensor that corresponds to the increments of the primary vectors.

4. Numerical Analysis of the Deformation of a Ring Transmission. Wave transmissions of the ring type — radial thin rings separated by a layer of rollers — are used in automatic drives. The outer surface of the external ring (the wave former) has a tooth-shaped profile. Precise design of these transmissions for prescribed quality factors requires analysis of the deformation of the ring elements during operation. Forces which are exerted by the rigid wheel and coupling cannot be reduced to a plane system and cause spatial deformation of the ring set.

For analysis of the operational deformations of a ring transmission, the latter is modeled by a layered inhomogeneous circular ring whose cross section A is shown in Fig. 1. We study small spatial deformations superposed on a planar uniformly compressed state of the ring. The analysis was performed using the variational equation (3.2). A linear finite-element approximation of the integrands was used. The number of nodes was varied within 1000–2000. Point action was approximated by an U-shaped function on length h of the corresponding cell.

Below, we give results of solution of two problems of the spatial deformation of a ring transmission subjected to a self-balanced nonplanar force system (Fig. 2). The structure consists of three circular rings 1–3 of constant thickness and an intermediate layer of cylindrical rollers 4 (Fig. 1). The dimensions of the rings are specified by the radii (in millimeters): $r_0 = 24.2$, $r_1 = 25.1$, $r_2 = 26$, $r_3 = 30.5$, and $R = r_4 = 31.3$ and the width $2l = 18$ mm equal for all the rings. The elastic properties of the rings are the same with Young's modulus $D_3 = 2.18 \cdot 10^5$ N/mm^2 and Poisson ratio $\gamma = 0.31$. The stiffness tensor in (2.6) is represented by the Hooke's matrix

$$[\boldsymbol{D}] = \begin{pmatrix} D_1 & 0 & 0 \\ 0 & D_2 & 0 \\ 0 & 0 & D_3 \end{pmatrix}, \qquad D_1 = D_2 = \frac{1}{2} D_3 (1+\gamma)^{-1}.$$

For the layer of rollers, $D_1 = D_2 = D_3 = 0$. The layered inhomogeneous structure of the ring is taken into account in calculating the generalized stiffnesses $\boldsymbol{E}$, $\boldsymbol{E}_i$, and $\boldsymbol{E}_{ij}$.

The first and second problems correspond to the schemes of loading of the ring transmission whose cells are subjected to point loads shown in Fig 2:

$$\{\Delta \boldsymbol{p}(\varphi_0)\} = \{\Delta \boldsymbol{p}(-\varphi_0)\} = \{0, -P/h, 0\}, \qquad \{\Delta \boldsymbol{q}(\varphi_0)\} = \{\Delta \boldsymbol{q}(-\varphi_0)\} = \{0, 0, -Pl/h\},$$

$$\{\Delta \boldsymbol{p}(\varphi_0)\} = \{\Delta \boldsymbol{p}(-\varphi_0)\} = \{0, -P/h, 0\}, \qquad \{-\Delta \boldsymbol{q}(\varphi_0)\} = \{\Delta \boldsymbol{q}(-\varphi_0)\} = \{0, 0, -Pl/h\}.$$

In the first problem, the self-balanced system of point forces is applied at one edge of the wave former, and in the second problem it is applied to both edges. In the middle parallel section of the ring, it reduces to the force vector $\Delta \boldsymbol{p}$ and the moment vector $\Delta \boldsymbol{q}$. The braces denote column vectors; the angular coordinate $\varphi = t_3/R$ is measured from the vertical diameter of the ring; the calculations were carried out for $\varphi_0 = \pi/20$ rad and $P = 10^4$ N.

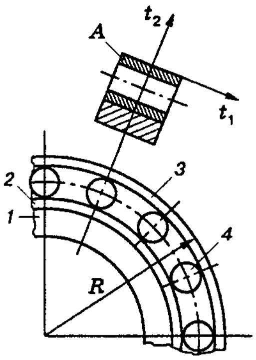

Fig. 1

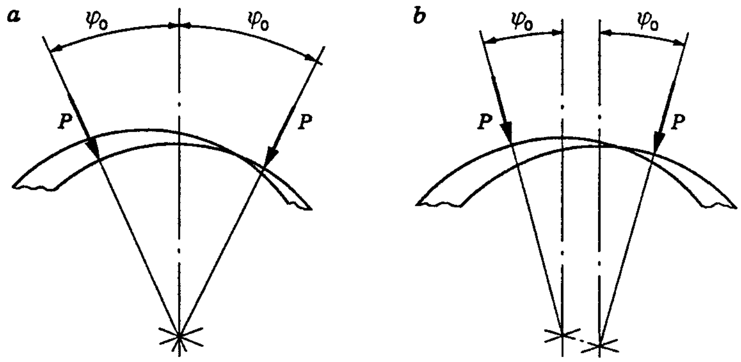

Fig. 2

TABLE 1

φ, rad	Problem 1				Problem 2			
	Δu_1, mm	Δu_2, mm	$\Delta\omega_2$, rad	$\Delta\omega_3$, rad	Δu_1, mm	Δu_2, mm	$\Delta\omega_2$, rad	$\Delta\omega_3$, rad
$-\pi/2$	−0.165	0.428	0	0.005	0	0.428	0	0
$-\pi/3$	−0.090	0.219	0.007	0	0.013	0.219	0.001	−0.001
$-\pi/6$	0.080	−0.240	0.008	−0.012	0.027	−0.240	0	−0.003
0	0.187	−0.512	0	−0.022	0	−0.512	−0.002	0
$\pi/6$	0.078	−0.242	−0.008	−0.012	−0.027	−0.242	0	0.003
$\pi/3$	0.090	0.218	−0.007	0	−0.013	0.218	0.001	0.001
$\pi/2$	−0.165	0.427	0	0.005	0	0.427	0	0

Table 1 shows the distributions of the additional displacements and rotations superimposed on the uniformly compressed state of the ring transmission over its outside perimeter (Δu_1 is the out-of-plane displacement, Δu_2 is the radial displacement, $\Delta\omega_2$ is the angle of rotation of the generatrix about the radius, and $\Delta\omega_3$ is the angle of torsion). In the first problem, the ring is bent out of the plane symmetrically about the horizontal diameter, and the function $\Delta\omega_2$ is antisymmetric. In the second problem, the ring is twisted about the vertical diameter, and the functions Δu_2 and $\Delta\omega_2$ are symmetric and the functions Δu_1 and $\Delta\omega_3$ are antisymmetric. The calculations were carried out on an IBM PC 386. The computational program has been used in designing wave transmissions of the ring type with specified quality factors.

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